# Sparse Identification of Nonlinear Dynamics (SINDy)

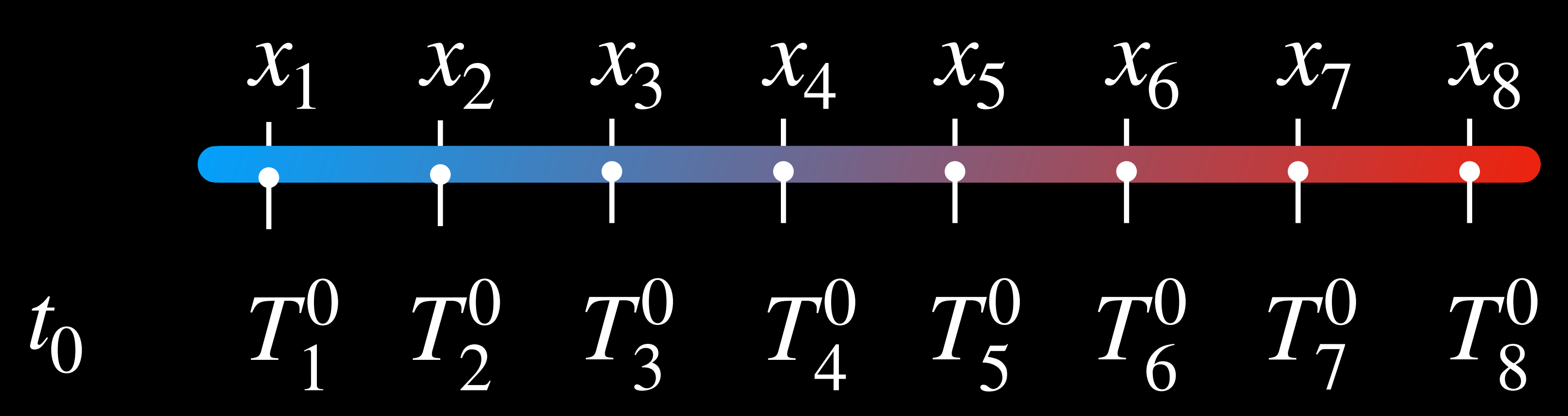
*Brunton, S. L., Proctor, J. L., & Kutz, J. N. (2016). PNAS*

# Sparse Identification of Nonlinear Dynamics (SINDy)

Heat diffusion through rod

# Sparse Identification of Nonlinear Dynamics (SINDy)

$$T_i^j = T(x_i, t_j)$$



# Sparse Identification of Nonlinear Dynamics (SINDy)

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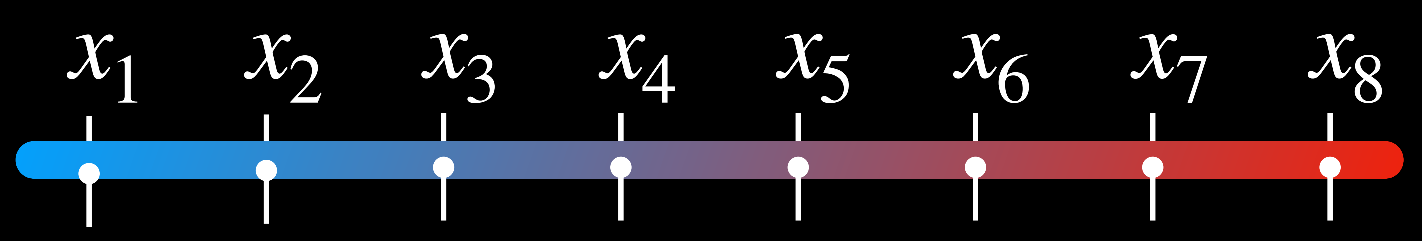


| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $t_0$ | $T_1^0$ | $T_2^0$ | $T_3^0$ | $T_4^0$ | $T_5^0$ | $T_6^0$ | $T_7^0$ | $T_8^0$ |
| $t_1$ | $T_1^1$ | $T_2^1$ | $T_3^1$ | $T_4^1$ | $T_5^1$ | $T_6^1$ | $T_7^1$ | $T_8^1$ |

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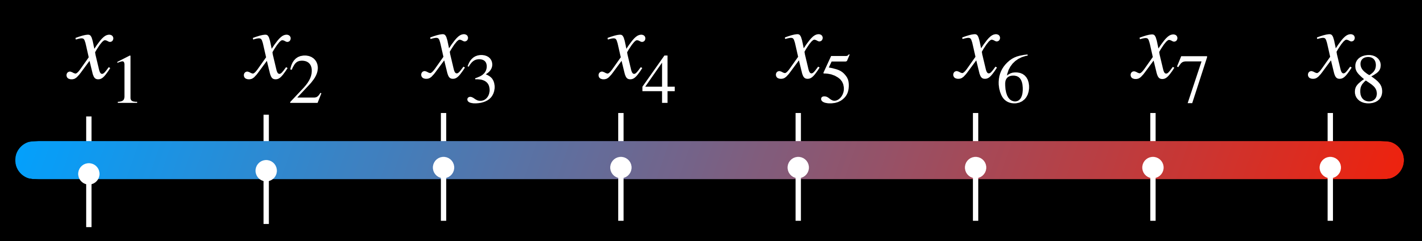


| | | | | | | | | |
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| $t_2$ | $T_1^2$ | $T_2^2$ | $T_3^2$ | $T_4^2$ | $T_5^2$ | $T_6^2$ | $T_7^2$ | $T_8^2$ |

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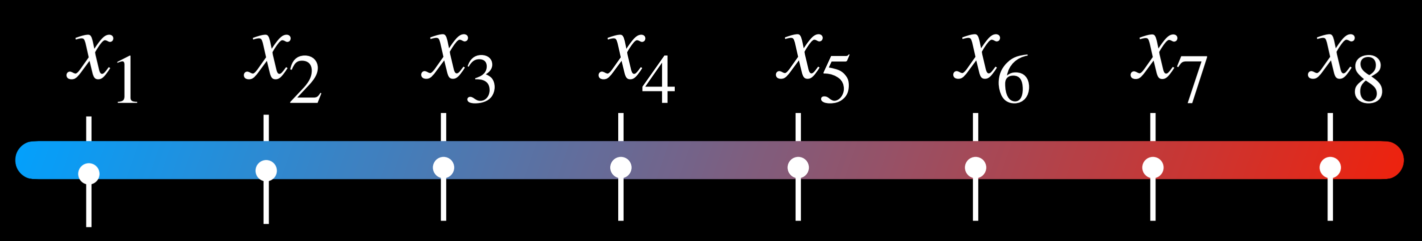


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$\vdots$

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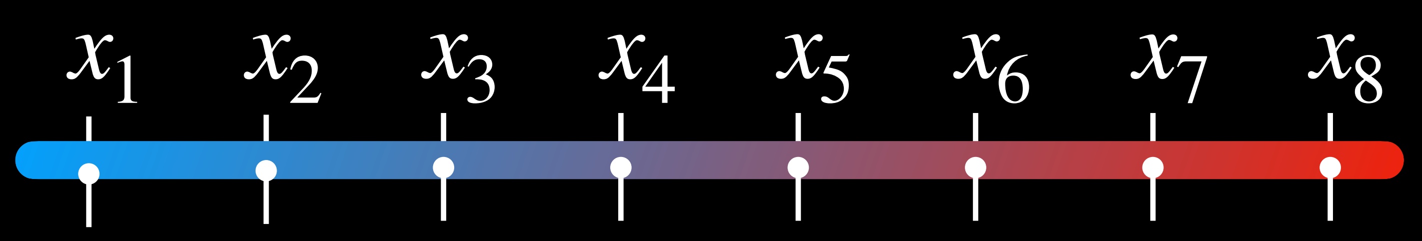


| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $t_0$ | $T_1^0$ | $T_2^0$ | $T_3^0$ | $T_4^0$ | $T_5^0$ | $T_6^0$ | $T_7^0$ | $T_8^0$ |
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## Assume the general form

$$\frac{\partial T}{\partial t} = f\left(\frac{\partial T}{\partial x}, \frac{\partial^2 T}{\partial x^2}, \ldots, T\frac{\partial T}{\partial x}\right)$$

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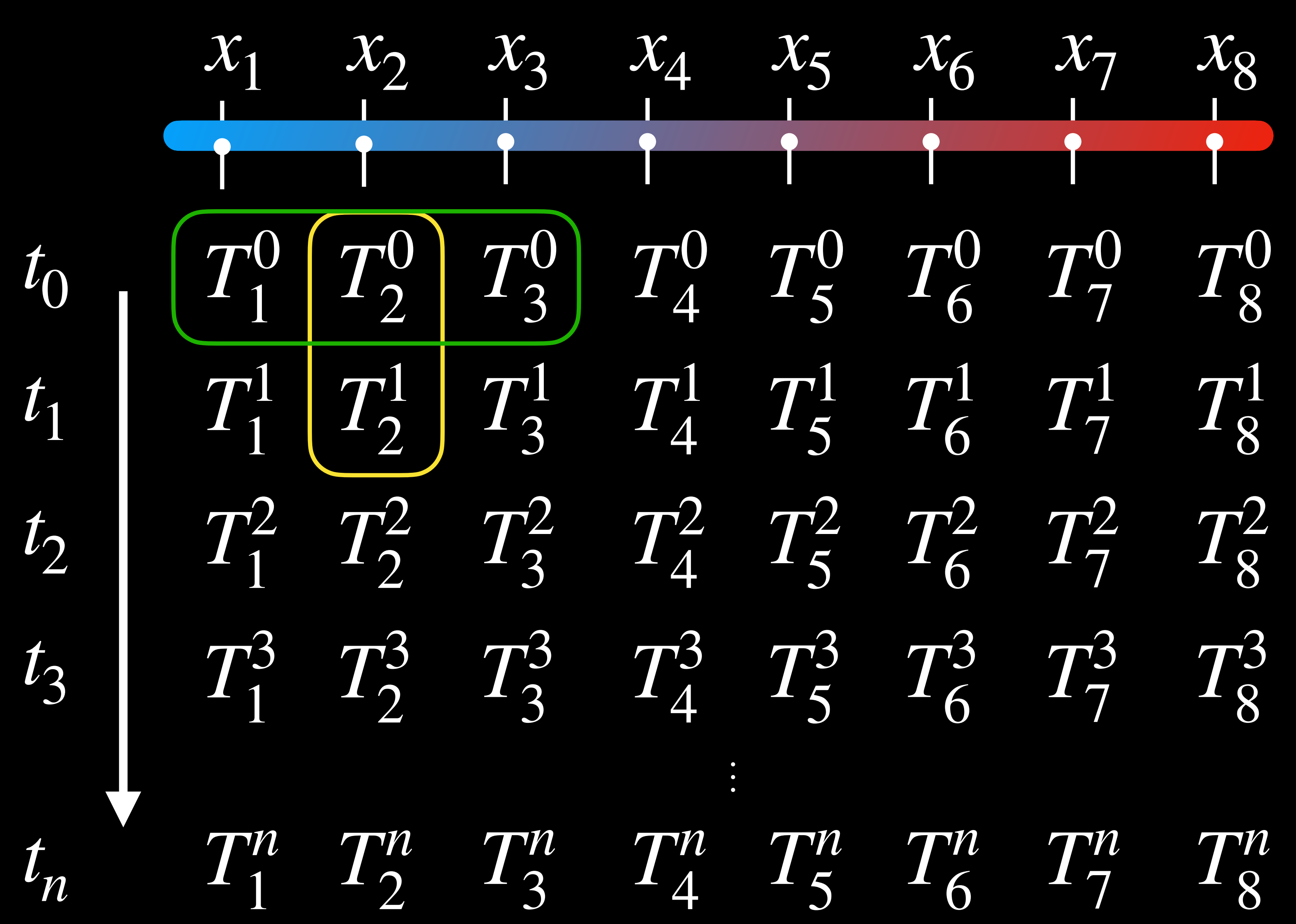


$$\frac{T_i^{j+1} - T_i^j}{t_{j+1} - t_j} \qquad \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{\Delta x^2} \qquad \frac{T_i^{j+1} - T_i^j}{x_{i+1} - x_i} \quad \cdots \quad T_i^j \frac{T_i^{j+1} - T_i^j}{x_{i+1} - x_i}$$

$$\left(\frac{\Delta T}{\Delta t}\right)_1 \qquad \left(\frac{\Delta^2 T}{\Delta x^2}\right)_1 \qquad \left(\frac{\Delta T}{\Delta x}\right)_1 \quad \cdots \quad T\left(\frac{\Delta T}{\Delta x}\right)_1$$

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$$\frac{T_i^{j+1} - T_i^j}{t_{j+1} - t_j} \qquad \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{\Delta x^2} \qquad \frac{T_i^{j+1} - T_i^j}{x_{i+1} - x_i} \qquad \cdots \qquad T_i^j \frac{T_i^{j+1} - T_i^j}{x_{i+1} - x_i}$$

$$\left(\frac{\Delta T}{\Delta t}\right)_1 \approx \left[ \left(\frac{\Delta^2 T}{\Delta x^2}\right)_1 \quad \left(\frac{\Delta T}{\Delta x}\right)_1 \quad \cdots \quad T\left(\frac{\Delta T}{\Delta x}\right)_1 \right] \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix}$$

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$$\begin{matrix} \left(\frac{\Delta T}{\Delta t}\right)_1 \\ \left(\frac{\Delta T}{\Delta t}\right)_2 \\ \vdots \end{matrix} \begin{matrix} \approx \\ \approx \\ \phantom{\vdots} \end{matrix} \begin{bmatrix} \left(\frac{\Delta^2 T}{\Delta x^2}\right)_1 & \left(\frac{\Delta T}{\Delta x}\right)_1 & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_1 \\ \left(\frac{\Delta^2 T}{\Delta x^2}\right)_2 & \left(\frac{\Delta T}{\Delta x}\right)_2 & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_2 \\ \vdots & \vdots & & \vdots \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix}$$

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$$\begin{matrix} \left(\frac{\Delta T}{\Delta t}\right)_1 \\ \left(\frac{\Delta T}{\Delta t}\right)_2 \\ \vdots \\ \left(\frac{\Delta T}{\Delta t}\right)_k \end{matrix} \begin{matrix} \approx \\ \approx \\ \\ \approx \end{matrix} \begin{bmatrix} \left(\frac{\Delta^2 T}{\Delta x^2}\right)_1 & \left(\frac{\Delta T}{\Delta x}\right)_1 & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_1 \\ \left(\frac{\Delta^2 T}{\Delta x^2}\right)_2 & \left(\frac{\Delta T}{\Delta x}\right)_2 & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_2 \\ \vdots & \vdots & & \vdots \\ \left(\frac{\Delta^2 T}{\Delta x^2}\right)_k & \left(\frac{\Delta T}{\Delta x}\right)_k & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_k \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix}$$

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$$\frac{\partial T}{\partial t} = f\left(\frac{\partial T}{\partial x}, \frac{\partial^2 T}{\partial x^2}, \ldots, T\frac{\partial T}{\partial x}\right)$$

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$$T_t \qquad \Theta \qquad \xi$$

## Assume the general form

$$\frac{\partial T}{\partial t} = f\left(\frac{\partial T}{\partial x}, \frac{\partial^2 T}{\partial x^2}, \ldots, T\frac{\partial T}{\partial x}\right)$$

## Objective

$$\hat{\xi} = \underset{\xi}{\operatorname{argmin}} \left\| T_t - \Theta\xi \right\|_2^2 + \lambda \left\| \xi \right\|_0$$

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$$\underbrace{\begin{pmatrix} \left(\frac{\Delta T}{\Delta t}\right)_1 \\ \left(\frac{\Delta T}{\Delta t}\right)_2 \\ \vdots \\ \left(\frac{\Delta T}{\Delta t}\right)_k \end{pmatrix}}_{T_t} \begin{matrix} \approx \\ \approx \\ \\ \approx \end{matrix} \underbrace{\begin{bmatrix} \left(\frac{\Delta^2 T}{\Delta x^2}\right)_1 & \left(\frac{\Delta T}{\Delta x}\right)_1 & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_1 \\ \left(\frac{\Delta^2 T}{\Delta x^2}\right)_2 & \left(\frac{\Delta T}{\Delta x}\right)_2 & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_2 \\ \vdots & \vdots & & \vdots \\ \left(\frac{\Delta^2 T}{\Delta x^2}\right)_k & \left(\frac{\Delta T}{\Delta x}\right)_k & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_k \end{bmatrix}}_{\Theta} \cdot \underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix}}_{\xi}$$

**Assume the general form**

$$\frac{\partial T}{\partial t} = f\left(\frac{\partial T}{\partial x}, \frac{\partial^2 T}{\partial x^2}, \ldots, T\frac{\partial T}{\partial x}\right)$$

**Objective**

$$\hat{\xi} = \underset{\xi}{\operatorname{argmin}} \left\|T_t - \Theta\xi\right\|_2^2 + \lambda \left\|\xi\right\|_0$$

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| $t_0$ | $T_1^0$ | $T_2^0$ | $T_3^0$ | $T_4^0$ | $T_5^0$ | $T_6^0$ | $T_7^0$ | $T_8^0$ |
| $t_1$ | $T_1^1$ | $T_2^1$ | $T_3^1$ | $T_4^1$ | $T_5^1$ | $T_6^1$ | $T_7^1$ | $T_8^1$ |
| $t_2$ | $T_1^2$ | $T_2^2$ | $T_3^2$ | $T_4^2$ | $T_5^2$ | $T_6^2$ | $T_7^2$ | $T_8^2$ |
| $t_3$ | $T_1^3$ | $T_2^3$ | $T_3^3$ | $T_4^3$ | $T_5^3$ | $T_6^3$ | $T_7^3$ | $T_8^3$ |
| | | | | $\vdots$ | | | | |
| $t_n$ | $T_1^n$ | $T_2^n$ | $T_3^n$ | $T_4^n$ | $T_5^n$ | $T_6^n$ | $T_7^n$ | $T_8^n$ |

$$\frac{T_i^{j+1} - T_i^j}{t_{j+1} - t_j} \qquad \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{\Delta x^2} \qquad \frac{T_i^{j+1} - T_i^j}{x_{i+1} - x_i} \quad \cdots \quad T_i^j \frac{T_i^{j+1} - T_i^j}{x_{i+1} - x_i}$$

$$\underbrace{\begin{pmatrix} \left(\frac{\Delta T}{\Delta t}\right)_1 \\ \left(\frac{\Delta T}{\Delta t}\right)_2 \\ \vdots \\ \left(\frac{\Delta T}{\Delta t}\right)_k \end{pmatrix}}_{T_t} \approx \underbrace{\begin{pmatrix} \left(\frac{\Delta^2 T}{\Delta x^2}\right)_1 & \left(\frac{\Delta T}{\Delta x}\right)_1 & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_1 \\ \left(\frac{\Delta^2 T}{\Delta x^2}\right)_2 & \left(\frac{\Delta T}{\Delta x}\right)_2 & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_2 \\ \vdots & \vdots & & \vdots \\ \left(\frac{\Delta^2 T}{\Delta x^2}\right)_k & \left(\frac{\Delta T}{\Delta x}\right)_k & \cdots & T\left(\frac{\Delta T}{\Delta x}\right)_k \end{pmatrix}}_{\Theta} \cdot \underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix}}_{\xi}$$

**We discover**

$$\frac{\partial T}{\partial t} = \xi_1 \frac{\partial^2 T}{\partial x^2}$$

**Objective**

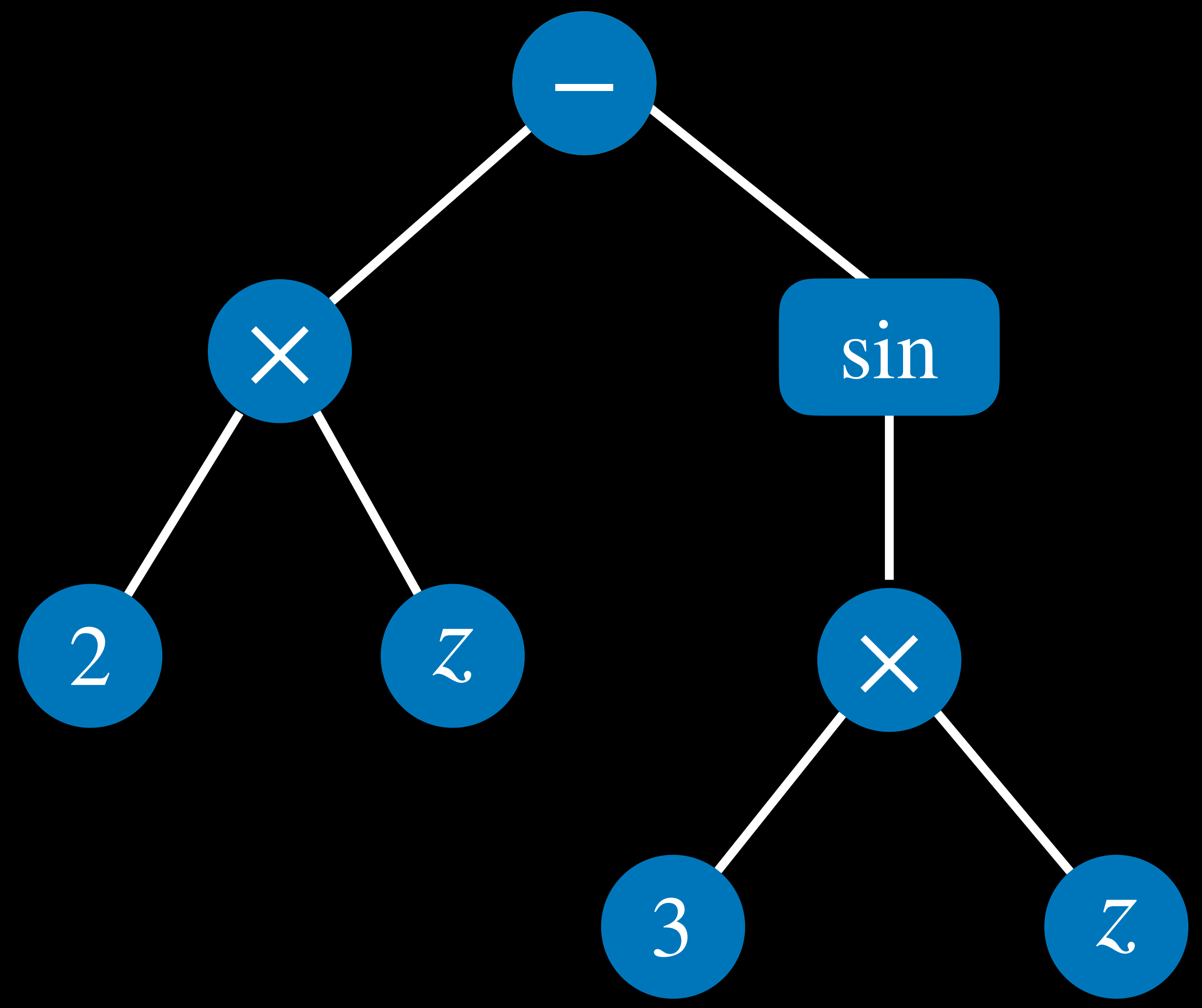
$$\hat{\xi} = \underset{\xi}{\operatorname{argmin}} \left\| T_t - \Theta \xi \right\|_2^2 + \lambda \left\| \xi \right\|_0$$

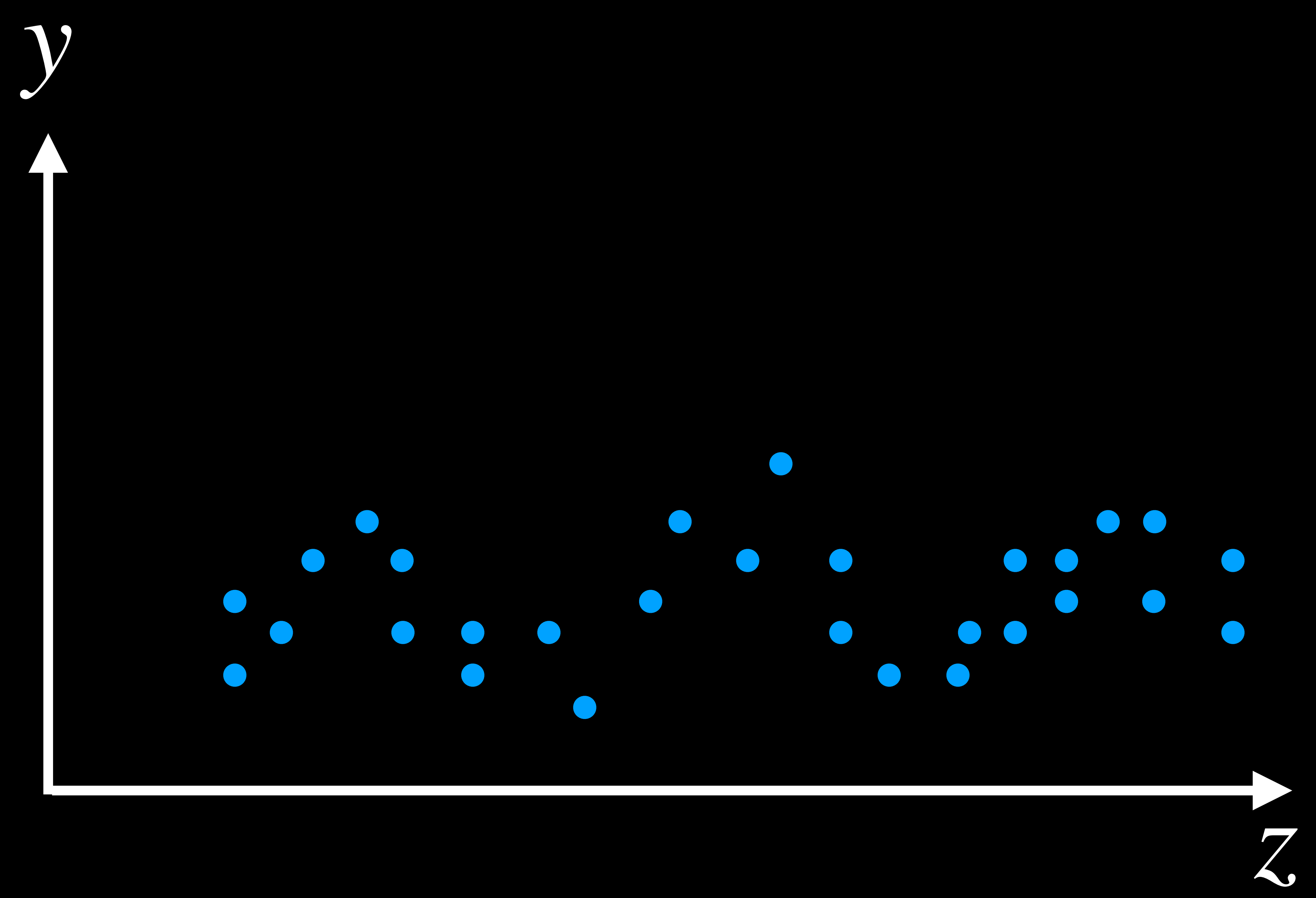
# Symbolic Regression

***BACON**, Patrick Langley (1977)*
***Eureqa**, Hod Lipson et. al (2009)*
***AI Feynman**, Max Tegmark et. al (2020)*

## Equations are trees

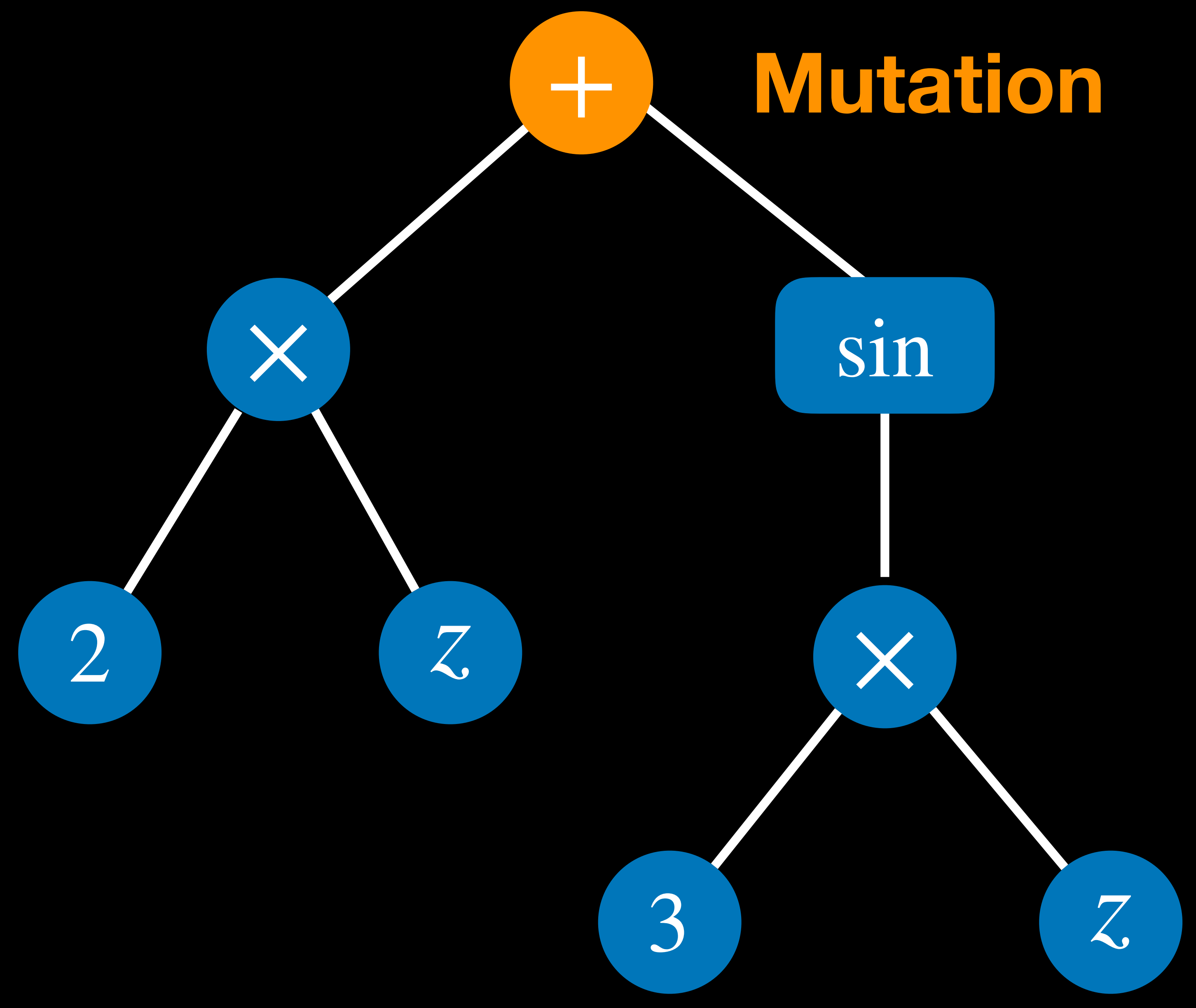
$$y = 2z - \sin(3z)$$

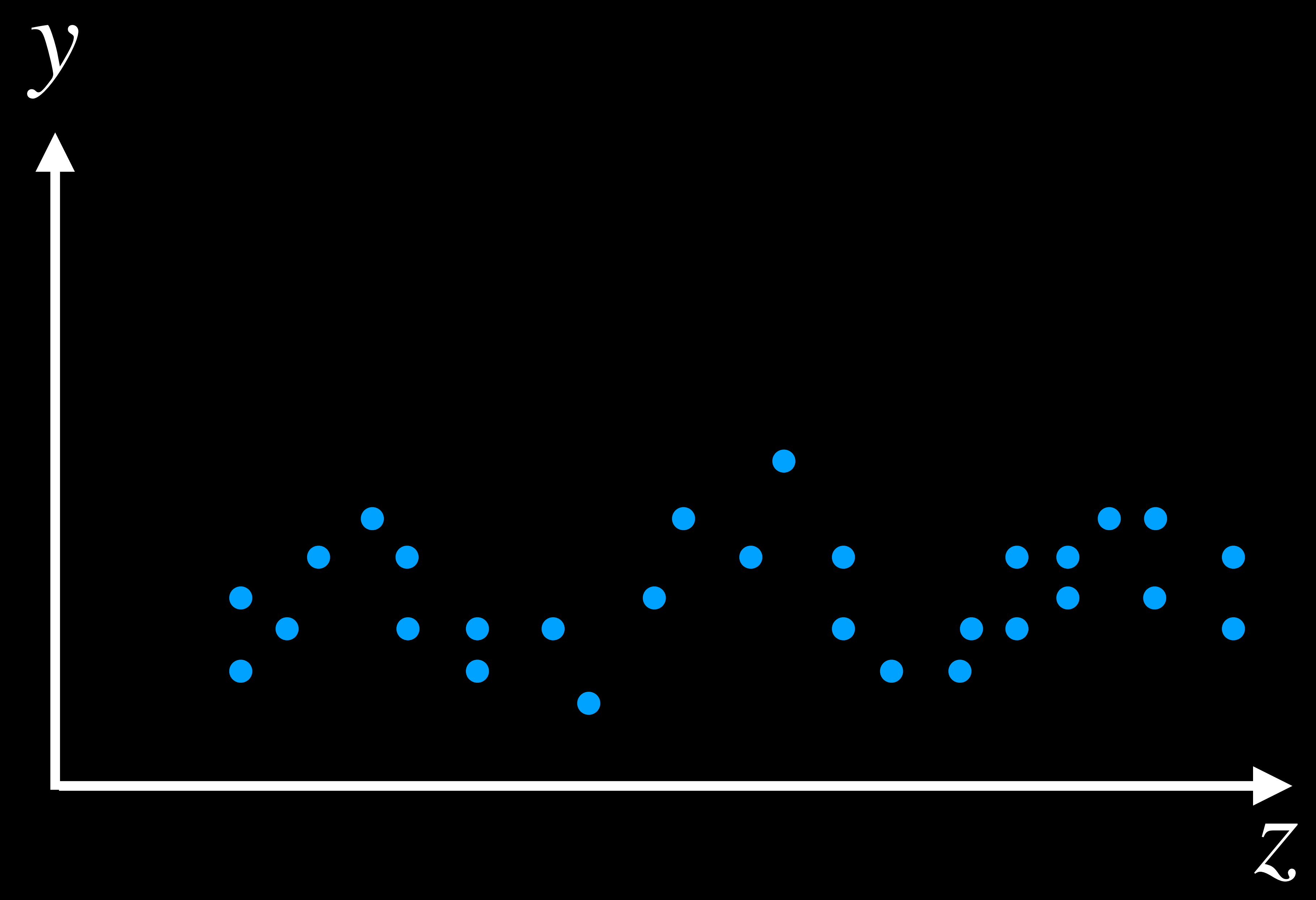


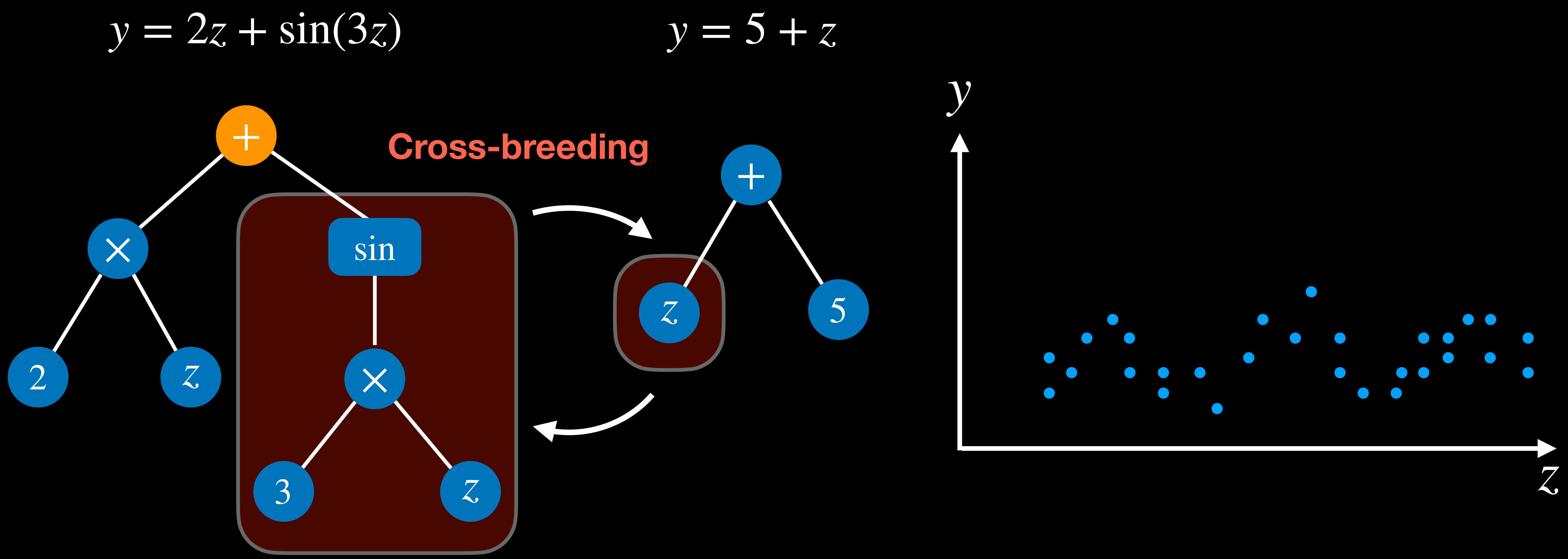
# Symbolic Regression

$$y = 2z + \sin(3z)$$



# Symbolic Regression



# Symbolic Regression